

Nonuniform Dichotomy Spectrum and Normal Forms for Nonautonomous Differential Systems

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Abstract

The aim of this paper is to study the normal forms of nonautonomous differential systems. For doing so, we first investigate the nonuniform dichotomy spectrum of the linear evolution operators that admit a nonuniform exponential dichotomy, where the linear evolution operators are defined by nonautonomous differential equations $\dot{x} = A(t)x$ in \mathbb{R}^n . Using the nonuniform dichotomy spectrum we obtain the normal forms of the nonautonomous linear differential equations. Finally we establish the finite jet normal forms of the nonlinear differential systems $\dot{x} = A(t)x + f(t, x)$ in \mathbb{R}^n , which is based on the nonuniform dichotomy spectrum and the normal forms of the nonautonomous linear systems.

Key words and phrases: Nonuniform exponential dichotomy, nonuniform dichotomy spectrum, nonautonomous differential system, normal form.

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1 Introduction and statement of the main results

The normal form theory in dynamical systems is to simplify ordinary differential equations through the change of variables. This theory can be traced back to Poincaré [21]. Some classical results in this direction for autonomous differential systems are the Poincaré–Dulac normal form theorem [22], the Siegel’s theorem [23], the Hartman–Grobman’s theorem [11, 12], the Sternberg’s theorem [27, 28], the Chen’s theorem [7], the Takens’ theorem [31] and so on. See also [6, 8, 13, 14, 33] and the recent survey paper [29] and the references

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therein. For nonautonomous systems, Barreira and Valls had established several results on the topological conjugacy between nonuniformly hyperbolic dynamical systems (see e.g. [2], [3]–[5]). Using the resonance of the dichotomy spectrum to study the normal forms of nonautonomous system, Siegmund [26] obtained a finite order normal form, and Wu and Li [32] got analytic normal forms of a class of analytic nonautonomous differential systems. As our knowledge, these last two papers are the only ones in which the normal forms of nonautonomous systems via the dichotomy spectrums were studied. Recently Li, Llibre and Wu [15] and [17] also had studied the normal forms of almost periodic differential and difference equations, respectively. For random differential systems there also appeared some results on normal forms [18, 19, 20], in which they extended the Poincaré’s, the Sternberg’s and the Siegel’s normal form theorems for autonomous differential systems to random dynamical systems.

As well-known, the normal form theory has played important roles in the study of bifurcation and some related topics of dynamical systems. Recently this theory has been successfully applied to study the embedding flow problem of diffeomorphisms, see for instance [16, 34, 35, 36].

In this paper we will study the normal forms of nonautonomous differential systems with their linear parts admitting a nonuniform exponential dichotomy. For this aim we first consider the nonautonomous linear differential systems in \mathbb{R}^n

$$\dot{x} = A(t)x, \quad (1.1)$$

with $A(t) \in M_n(\mathbb{R})$ the set of square matrix functions of n th order defined in \mathbb{R} , we assume in this paper that each solution of system (1.1) is defined on the whole \mathbb{R} . Denote by $\Phi(t, s)$ the evolution operator associated to system (1.1). Then we have

$$x(t) = \Phi(t, s)x(s), \quad \Phi(t, s)\Phi(s, \tau) = \Phi(t, \tau) \text{ for all } t, s, \tau \in \mathbb{R},$$

where $x(t)$ is a solution of system (1.1).

We say that system (1.1) admits a *nonuniform exponential dichotomy* if there exists an invariant projection $P(t) \in M_n(\mathbb{R})$ (where *invariant* means that $P(t)\Phi(t, s) = \Phi(t, s)P(s)$ for all $t, s \in \mathbb{R}$), and $K \geq 1$, $\alpha < 0 < \beta$ and $\mu, \nu \geq 0$ with $\alpha + \mu < 0$, $\beta - \nu > 0$ and $\max\{\mu, \nu\} \leq \min\{-\alpha, \beta\}$ such that

$$\begin{aligned} \|\Phi(t, s)P(s)\| &\leq Ke^{\alpha(t-s)+\mu|s|} \text{ for } t \geq s, \\ \|\Phi(t, s)(I - P(s))\| &\leq Ke^{\beta(t-s)+\nu|s|} \text{ for } t \leq s. \end{aligned}$$

If $\mu = \nu = 0$ it defines the uniform exponential dichotomy or simply exponential dichotomy (see e.g. [24, 9]). Barreira and Valls [4] showed that the system in \mathbb{R}^2

$$\dot{x} = -(\omega + at \sin t)x, \quad \dot{y} = (\omega + at \sin t)y,$$

admits a nonuniform exponential dichotomy but does not admit a uniform exponential dichotomy.

In our definition of the nonuniform exponential dichotomy there appear the extra conditions $\alpha + \mu < 0$, $\beta - \nu > 0$ and $\max\{\mu, \nu\} \leq \min\{-\alpha, \beta\}$, which did not appear explicitly in the definition of [2, 3, 5]. In fact, in their results on the conjugacy between two nonautonomous dynamical systems they always assume that the nonuniform constants μ and ν are sufficiently small, and consequently the extra conditions hold implicitly.

The *nonuniform dichotomy spectrum* of system (1.1) is the set

$$\Sigma(A) = \{\gamma \in \mathbb{R}; \dot{x} = (A(t) - \gamma I)x \text{ admits no nonuniform exponential dichotomy}\}.$$

Its complement $\rho(A) = \mathbb{R} \setminus \Sigma(A)$ is called the *resolvent set* of system (1.1).

A *linear integral manifold* of system (1.1) is a nonempty set W of $\mathbb{R} \times \mathbb{R}^n$ satisfying $\{(t, \Phi(t, \tau)\xi); t \in \mathbb{R}\} \subset W$ for each $(\tau, \xi) \in W$, and for any given $\tau \in \mathbb{R}$ the fiber $W(\tau) = \{\xi \in \mathbb{R}^n; (\tau, \xi) \in W\}$ is a linear subspace of \mathbb{R}^n . In the following we also call W *invariant* by (1.1). We note that all the fibers $W(\tau)$ have the same dimension, denoted by $\dim W$, and they form a vector bundle over \mathbb{R} . A linear integral manifold is a topological manifold in $\mathbb{R} \times \mathbb{R}^n$.

Let W_1 and W_2 be two linear integral manifolds of (1.1). Their *intersection* and *sum* are defined respectively as

$$\begin{aligned} W_1 \cap W_2 &= \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n; \xi \in W_1(\tau) \cap W_2(\tau)\}, \\ W_1 + W_2 &= \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n; \xi \in W_1(\tau) + W_2(\tau)\}. \end{aligned}$$

They are also linear integral manifolds. A sum of linear integral manifolds W_1, \dots, W_k is called *Whitney-sum*, denoted by $W_1 \oplus \dots \oplus W_k$, if $W_i \cap W_j = \mathbb{R} \times \{0\}$ for $1 \leq i \neq j \leq k$.

For a $\gamma \in \mathbb{R}$ we define two subsets of $\mathbb{R} \times \mathbb{R}^n$:

$$\begin{aligned} \mathcal{U}_\gamma &= \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n; \sup_{t \geq 0} \|\Phi(t, \tau)\xi\| e^{-\gamma t} < \infty \right\}, \\ \mathcal{V}_\gamma &= \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n; \sup_{t \leq 0} \|\Phi(t, \tau)\xi\| e^{-\gamma t} < \infty \right\}. \end{aligned} \tag{1.2}$$

In this paper the notations \mathcal{U}_γ and \mathcal{V}_γ always denote the sets defined in (1.2), respectively.

Our first result is on the structure of the nonuniform dichotomy spectrum of system (1.1). It is the generalization of the spectral theorem of [24] for the dichotomy spectrum to the nonuniform dichotomy spectrum of system (1.1).

Theorem 1.1. *For system (1.1), the following statements hold.*

- (a) *The nonuniform dichotomy spectrum $\Sigma(A)$ of system (1.1) is the union of m disjoint closed intervals in \mathbb{R} (called spectral intervals) with $0 \leq m \leq n$. Precisely, if $m = 0$ then $\Sigma(A) = \emptyset$; if $m = 1$ then $\Sigma(A) = \mathbb{R}$ or $(-\infty, b_1]$ or $[a_1, b_1]$ or $[a_1, \infty)$; if $m > 1$ then $\Sigma(A) = I_1 \cup [a_2, b_2] \cup \dots \cup [a_{m-1}, b_{m-1}] \cup I_m$ with $I_1 = [a_1, b_1]$ or $(-\infty, b_1]$ and $I_m = [a_m, b_m]$ or $[a_m, \infty)$, where $a_i \leq b_i < a_{i+1}$ for $i = 1, \dots, m-1$.*

(b) If $m \geq 1$ and $\Sigma(A) \neq \mathbb{R}$, assume that

$$\Sigma(A) = I_1 \cup [a_2, b_2] \cup \dots \cup [a_{m-1}, b_{m-1}] \cup I_m,$$

with I_1, I_m and a_i, b_i given in statement (a). If $I_1 = [a_1, b_1]$ and $I_m = [a_m, b_m]$, set $b_0 = -\infty$ and $a_{m+1} = \infty$, and choose $\gamma_i \in (b_i, a_{i+1})$ for $i = 0, 1, \dots, m$, we have the linear integral manifolds \mathcal{U}_{γ_i} and \mathcal{V}_{γ_i} for $i = 0, 1, \dots, m$. If $I_1 = (-\infty, b_1]$, we choose $\gamma_0 < b_1$ and set $\mathcal{U}_{\gamma_0} = \mathbb{R} \times \{0\}$ and $\mathcal{V}_{\gamma_0} = \mathbb{R} \times \mathbb{R}^n$. If $I_m = [a_m, \infty)$, we choose $\gamma_m > a_m$ and set $\mathcal{U}_{\gamma_m} = \mathbb{R} \times \mathbb{R}^n$ and $\mathcal{V}_{\gamma_m} = \mathbb{R} \times \{0\}$. Define

$$\mathcal{W}_0 = \mathcal{U}_{\gamma_0}, \quad \mathcal{W}_i = \mathcal{U}_{\gamma_i} \cap \mathcal{V}_{\gamma_{i-1}} \text{ for } i = 1, \dots, m, \quad \mathcal{W}_{m+1} = \mathcal{V}_{\gamma_m}.$$

Then $\dim \mathcal{W}_i \geq 1$ for $i = 1, \dots, m$ and

$$\mathbb{R} \times \mathbb{R}^n = \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \dots \oplus \mathcal{W}_{m+1}.$$

The linear integral manifold \mathcal{W}_i is called a *spectral manifold* for $i = 0, \dots, m+1$. We shall see from Proposition 2.3 below that the spectral manifold \mathcal{W}_i is independent of the choice of γ_i .

Next we present a sufficient condition for a nonuniform dichotomy spectrum to be nonempty and bounded.

The evolution operator $\Phi(t, s)$ of $\dot{x} = A(t)x$ has a *nonuniformly bounded growth* if there exist $K \geq 1$, $a \geq 0$ and $\varepsilon \geq 0$ such that

$$\|\Phi(t, s)\| \leq K e^{a|t-s| + \varepsilon|s|}, \quad t, s \in \mathbb{R}. \quad (1.3)$$

If $\varepsilon = 0$ the evolution operator has the so-called bounded growth (see [24])

Theorem 1.2. *Assume that the evolution operator of system (1.1) has a nonuniformly bounded growth. The following statements hold.*

- (a) *The nonuniform dichotomy spectrum $\Sigma(A)$ of system (1.1) is nonempty and bounded, i.e., $\Sigma(A) = [a_1, b_1] \cup \dots \cup [a_m, b_m]$ with $m \geq 1$ and $-\infty < a_1 \leq b_1 < \dots < a_m \leq b_m < \infty$.*
- (b) *$\mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \dots \oplus \mathcal{W}_m \oplus \mathcal{W}_{m+1} = \mathbb{R} \times \mathbb{R}^n$, where \mathcal{W}_i 's are the spectral manifolds defined in Theorem 1.1.*

For autonomous linear systems in \mathbb{R}^n it is well known that they can be transformed into normal forms with their coefficient matrices in the Jordan type through some nondegenerate linear changes of variables. Using the dichotomy spectrum Siegmund [25] provided a method to study the normal forms of nonautonomous linear systems. Here we extend his method to study the normal form of nonautonomous linear differential system using the nonuniform dichotomy spectrum.

As first defined in [10], we say that system (1.1) and the system

$$\dot{y} = B(t)y, \quad (1.4)$$

are *nonuniformly kinematically similar* if there exists a differentiable matrix function $S : \mathbb{R} \rightarrow \text{GL}_n(\mathbb{R})$ satisfying

$$\|S(t)\| \leq M_\varepsilon e^{\varepsilon|t|}, \quad \|S(t)^{-1}\| \leq M_\varepsilon e^{\varepsilon|t|}, \quad \text{for all } t \in \mathbb{R}, \quad (1.5)$$

with $M_\varepsilon > 0$ a constant, such that $x(t) = S(t)y(t)$ transforms (1.1) into (1.4). Correspondingly, the $S(t)$ satisfying (1.5) is called a *nonuniform Lyapunov matrix*, and the change of variables $x(t) = S(t)y(t)$ is a *nonuniform Lyapunov transformation*.

The following result characterizes the normal forms of nonautonomous linear differential systems via their nonuniform dichotomy spectrums.

Theorem 1.3. *Assume that $A(t)$ is differentiable, and that the evolution operator of system (1.1) has a nonuniformly bounded growth. Let $\Sigma(A) = [a_1, b_1] \cup \dots \cup [a_m, b_m]$ with $-\infty < a_1 \leq b_1 < \dots < a_m \leq b_m < \infty$ be the nonuniform dichotomy spectrum. Then system (1.1) is nonuniformly kinematically similar to*

$$\dot{y} = \begin{pmatrix} B_0(t) & & & & \\ & B_1(t) & & & \\ & & \ddots & & \\ & & & B_m(t) & \\ & & & & B_{m+1}(t) \end{pmatrix} y, \quad (1.6)$$

where $B_i(t) : \mathbb{R} \rightarrow \mathbb{R}^{n_i \times n_i}$ are differentiable with $n_i = \dim \mathcal{W}_i$, $\Sigma(B_0) = \Sigma(B_{m+1}) = \emptyset$ and $\Sigma(B_i) = [a_i, b_i]$ for $i = 1, \dots, m$. Recall that $\mathcal{W}_0, \mathcal{W}_1, \dots, \mathcal{W}_m, \mathcal{W}_{m+1}$ are the corresponding spectral manifolds.

Now we use the nonuniform dichotomy spectrums and the normal forms for nonautonomous linear differential systems to study the normal forms of nonautonomous nonlinear differential systems.

Consider the nonautonomous nonlinear differential system

$$\dot{x} = A(t)x + f(t, x), \quad x \in (\mathbb{R}^n, 0), \quad (1.7)$$

where $f(t, x) = O(|x|^2)$ is an analytic function.

Assume that the evolution operator of the linear system $\dot{x} = A(t)x$ has a nonuniformly bounded growth. Then its nonuniform dichotomy spectrum is $\Sigma(A) = [a_1, b_1] \cup \dots \cup [a_m, b_m]$ with $m \geq 1$ and $-\infty < a_1 \leq b_1 < \dots < a_m \leq b_m < \infty$. Hence by Theorem 1.3 system (1.7) is equivalent to

$$\dot{x}_i = A_i(t)x_i + f_i(t, x), \quad i = 1, \dots, m, \quad (1.8)$$

where A_i is an $n_i \times n_i$ matrix with $n_1 + \dots + n_m = n$ and $\Sigma(A_i) = [a_i, b_i]$ for $i = 1, \dots, m$. So there exist $\varepsilon_i > 0$ suitably small such that for $\sigma_i \in [a_i - \varepsilon_i, a_i]$ and $\rho_i \in (b_i, b_i + \varepsilon_i]$ systems $\dot{z} = (A_i(t) - \sigma_i I)z$ and $\dot{z} = (A_i(t) - \rho_i I)z$ admit nonuniform exponential dichotomies. Hence there exist $K_i \geq 1$, $\alpha_i < 0$, $\beta_i > 0$, and $\mu_i, \nu_i \geq 0$ with $\alpha_i + \mu_i < 0$ and $\beta_i - \nu_i > 0$ such that

$$\begin{aligned} \|\Phi_{A_i}(t, s)\| &\leq K_i e^{\rho_i(t-s)} e^{\alpha_i(t-s) + \mu_i|s|} \quad \text{for } t \geq s, \\ \|\Phi_{A_i}(t, s)\| &\leq K_i e^{\sigma_i(t-s)} e^{\beta_i(t-s) + \nu_i|s|} \quad \text{for } t \leq s. \end{aligned} \tag{1.9}$$

In what follows we study only system (1.8). Expanding $f_i(t, x)$ in the Taylor series

$$f_i(t, x) \sim \sum_{|l|=2}^{\infty} f_{il}(t) x^l, \quad i = 1, \dots, m,$$

where $l = (l_1, \dots, l_n) \in \mathbb{Z}_+^n$ are multiple indices with $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $x^l = x_1^{l_1} \dots x_n^{l_n}$ and $|l| = l_1 + \dots + l_n$.

In (1.8) a monomial, say $f_{il}(t)x^l$, is *nonresonant* if

$$[a_i, b_i] \cap \sum_{j=1}^m \tau_j [a_j, b_j] = \emptyset,$$

where the sum and the multiplication of intervals are defined as

$$[a, b] + [c, d] = [a + c, b + d], \quad k[a, b] = [ka, kb],$$

and $\tau = (\tau_1, \dots, \tau_m)$ is the image of $l = (l_1, \dots, l_n) \in \mathbb{Z}_+^n$ under the mapping

$$\begin{aligned} \mathcal{N} : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ l &\longrightarrow \tau = (l_1 + \dots + l_{n_1}, l_{n_1+1} + \dots + l_{n_1+n_2}, \dots, l_{n_1+n_2} + \dots + l_n). \end{aligned}$$

The notion nonresonance for nonautonomous differential systems is an extension of the one for autonomous system $\dot{x} = Ax + f(x)$, where the nonresonant condition is

$$\lambda_i \neq \sum_{j=1}^n k_j \lambda_j, \quad k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n, \quad |k| \geq 2,$$

with $\lambda = (\lambda_1, \dots, \lambda_n)$ the eigenvalues of the constant matrix A .

We say that system (1.8) is in the *normal form* if its nonlinear terms are all resonant. The transformation sending (1.8) to its normal form is called a *normalization*. Usually the normalization is not unique. If the normalization contains only nonresonant terms, then it is called a *distinguished normalization*. The corresponding normal form system is called in the *distinguished normal form*. We note that for a given differential system the Taylor expansion of its distinguished normalization is unique. Of course, if the distinguished normalization is analytic, then itself is unique.

Let $f(t, x) = (f_1(t, x), \dots, f_m(t, x))$ have the Taylor expansion $f_j(t, x) \sim \sum_{s=2}^{\infty} \tilde{f}_{js}(t, x)$, where \tilde{f}_{js} is a vector-valued homogeneous polynomial of degree s in x with its coefficients being the functions of t .

Theorem 1.4. *Assume that system (1.8) is analytic or C^∞ and that the evolution operator of the linear system associated with (1.8) has a nonuniformly bounded growth. Let $\Sigma(A) = [a_1, b_1] \cup \dots \cup [a_m, b_m]$ with $-\infty < a_1 \leq b_1 < \dots < a_m \leq b_m < \infty$ be the nonuniform dichotomy spectrum, and let α_i, β_i, μ_i and ν_i be the data defined in (1.9). Set $\varrho = \max\{\mu_i, \nu_i; i = 1, \dots, m\}$, and $\sigma = \min\{-\alpha_j, \beta_j, j = 1, \dots, m\}$. If $\sigma/\varrho > 4$ and there exists a positive number $k \in (3, \sigma/\varrho)$ such that the coefficient vectors of $\tilde{f}_s = (\tilde{f}_{1s}, \dots, \tilde{f}_{ms})$ according to the base $\{x^\tau e_j : \tau \in \mathbb{Z}_+^n, |\tau| = s, j = 1, \dots, n\}$, denoted by $p_s(t)$, satisfy*

$$\|p_s(t)\| \leq d_s \exp \left(- \left((s-1)k - \frac{(s+3)(s-2)}{2} \right) \varrho |t| \right), \quad \text{for } 2 \leq s < 2k-4, \quad (1.10)$$

then there exists a near identity polynomial map of degree $2k-5$ under which system (1.8) is transformed into

$$\dot{y} = Ay + g(t, y) + h(t, y), \quad (1.11)$$

where $g(t, y)$ consists of the resonant homogeneous polynomials in y of degrees from 2 to $2k-5$ with coefficients being the functions of t , which are uniformly convergent to zero when $|t| \rightarrow \infty$, and $h(t, y) = O(|y|^{2k-4})$.

In the last theorem we have several restricted conditions. We should say that except the one on the modulus $\|p_s(t)\|$, the others are natural. For instance $\sigma/\varrho > 4$ holds provided that the nonuniform exponents μ_i, ν_i are sufficiently small. The condition on the modulus $\|p_s(t)\|$ is also natural in some sense, because if $\|p_s(t)\|$ increases too fast as $|t|$ increases, any orbit starting in a small neighborhood of the origin will rapidly leave the neighborhood, and so the theorem will not be correct. If $\varrho = 0$ we are in the case of the uniform dichotomy spectrum.

We mention that if an analytic or a C^∞ system (1.8) has its linear part satisfying a nonuniform exponential dichotomy, it is nearly impossible to get an analytic or a C^∞ normalization which transforms system (1.8) to its normal form (of course, if system (1.8) is a polynomial one, the normalization may exist). Also Theorem 1.4 holds for C^{2k-4} differential systems. These can be seen from the proof of Theorem 1.4.

We also mention that even for a C^k ($2 < k < \infty$) smooth autonomous differential system of form (1.8), if $n > 2$ there is no satisfactory results on the regularity of the normalization which transforms system (1.8) to a polynomial normal form. For $n = 2$ this problem was solved by Stowe [30].

This paper is organized as follows. In the next section we shall prove Theorems 1.1 and 1.2. The proof of Theorems 1.3 and 1.4 will be given in Sections 3 and 4, respectively.

2 Proof of Theorems 1.1 and 1.2

For proving Theorems 1.1 and 1.2 we need some basic results which characterize the nonuniform dichotomy spectrum. The ideas of the proofs partially follow from [24].

2.1 The basic results

This subsection is a preparation for proving Theorems 1.1 and 1.2.

Proposition 2.1. *Let $\mathcal{U}_\gamma, \mathcal{V}_\gamma$ be the subsets of $\mathbb{R} \times \mathbb{R}^n$ defined in (1.2). The following statements hold.*

- (i) \mathcal{U}_γ and \mathcal{V}_γ are linear integral manifolds of system (1.1)
- (ii) If $\gamma_1 \leq \gamma_2$ then $\mathcal{U}_{\gamma_1} \subseteq \mathcal{U}_{\gamma_2}$ and $\mathcal{V}_{\gamma_1} \supseteq \mathcal{V}_{\gamma_2}$.

Proof. (i) For any $(\tau, \xi) \in \mathcal{U}_\gamma$, by definition we only need to prove $(s, \Phi(s, \tau)\xi) \in \mathcal{U}_\gamma$ for all $s \in \mathbb{R}$. In fact, it follows from the fact that

$$\sup_{t \geq 0} \|\Phi(t, s)\Phi(s, \tau)\xi\|e^{-\gamma t} = \sup_{t \geq 0} \|\Phi(t, \tau)\xi\|e^{-\gamma t} < \infty.$$

The proof for \mathcal{V}_γ follows from the same arguments as those for \mathcal{U}_γ .

(ii) The claim $\mathcal{U}_{\gamma_1} \subseteq \mathcal{U}_{\gamma_2}$ follows easily from $-\gamma_1 t \geq -\gamma_2 t$ for $t \geq 0$. A similar argument works with \mathcal{V}_γ . \square

Proposition 2.2. *For $\gamma \in \mathbb{R}$, if*

$$\dot{x} = (A(t) - \gamma I)x, \tag{2.1}$$

admits a nonuniform exponential dichotomy with an invariant projection P , then we have

$$\mathcal{U}_\gamma = \text{Im}P, \quad \mathcal{V}_\gamma = \text{Ker}P \quad \text{and} \quad \mathcal{U}_\gamma \oplus \mathcal{V}_\gamma = \mathbb{R} \times \mathbb{R}^n,$$

where $\text{Im}P$ and $\text{Ker}P$ denote the image and kernel of the projection P , respectively.

Proof. Let $\Phi(t, s)$ be the evolution operator of $\dot{x} = A(t)x$. Some easy calculations show that $\Phi_\gamma(t, s) = e^{-\gamma(t-s)}\Phi(t, s)$ is an evolution operator of (2.1), and that $P(t)$ is an invariant projection of $\Phi(t, \tau)$ if and only if it is an invariant projection of $\Phi_\gamma(t, \tau)$. By the assumption there exist $K_\gamma \geq 1$, $\alpha_\gamma < 0, \beta_\gamma > 0$ and $\mu_\gamma, \nu_\gamma \geq 0$ with $\alpha + \mu_\gamma < 0$ and $\beta - \nu_\gamma > 0$ such that

$$\begin{aligned} \|\Phi_\gamma(t, s)P(s)\| &\leq K_\gamma e^{\alpha_\gamma(t-s) + \mu_\gamma|s|} \text{ for all } t \geq s, \\ \|\Phi_\gamma(t, s)(I - P(s))\| &\leq K_\gamma e^{\beta_\gamma(t-s) + \nu_\gamma|s|} \text{ for all } t \leq s. \end{aligned}$$

First we prove $\mathcal{U}_\gamma \subset \text{Im}P$. For any $(\tau, \xi) \in \mathcal{U}_\gamma$, by definition there exists a constant c_γ such that

$$\|\Phi(t, \tau)\xi\| \leq c_\gamma e^{\gamma t} \text{ for all } t \geq 0.$$

It follows that

$$\|\Phi_\gamma(t, \tau)\xi\| = e^{-\gamma(t-\tau)}\|\Phi(t, \tau)\xi\| \leq c_\gamma e^{\gamma\tau} \text{ for all } t \geq 0.$$

Set $\xi = \xi_1 + \xi_2$ with $\xi_1 \in \text{Im}P(\tau)$ and $\xi_2 \in \text{Ker}P(\tau)$. Since $P(t)\Phi_\gamma(t, \tau) = \Phi_\gamma(t, \tau)P(\tau)$, we have

$$\xi_2 = (I - P(\tau))\xi = \Phi_\gamma(\tau, t)(I - P(t))\Phi_\gamma(t, \tau)\xi.$$

These yield that for $t \geq \max\{0, \tau\}$

$$\|\xi_2\| = K_\gamma e^{\beta_\gamma(\tau-t)+\nu_\gamma|t|}\|\Phi_\gamma(t, \tau)\xi\| \leq K_\gamma c_\gamma e^{-(\beta_\gamma-\nu_\gamma)t+(\beta_\gamma+\gamma)\tau}.$$

Hence we have $\xi_2 = 0$ because $\beta_\gamma - \nu_\gamma > 0$, and consequently $\xi = \xi_1 \in \text{Im}P(\tau)$. This proves that $\mathcal{U}_\gamma \subset \text{Im}P$.

For proving $\text{Im}P \subset \mathcal{U}_\gamma$, we assume that $\tau \in \mathbb{R}$, $\xi \in \text{Im}P(\tau)$. Then $P(\tau)\xi = \xi$. For $t \geq \max\{0, \tau\}$ we have

$$\|\Phi(t, \tau)\xi\|e^{-\gamma t} = e^{-\gamma\tau}\|\Phi_\gamma(t, \tau)P(\tau)\xi\| \leq K_\gamma e^{\alpha_\gamma t - (\gamma+\alpha_\gamma)\tau + \mu_\gamma|\tau|}.$$

This implies that $(\tau, \xi) \in \mathcal{U}_\gamma$ because $\alpha_\gamma < 0$, and so $\text{Im}P \subset \mathcal{U}_\gamma$. This proves that $\text{Im}P = \mathcal{U}_\gamma$.

Similarly using the assumption $\alpha_\gamma + \mu_\gamma < 0$ we can prove that $\mathcal{V}_\gamma = \text{Ker}P$. Finally the equality $\mathcal{U}_\gamma \oplus \mathcal{V}_\gamma = \mathbb{R} \times \mathbb{R}^n$ follows from $\mathcal{U}_\gamma = \text{Im}P$ and $\mathcal{V}_\gamma = \text{Ker}P$. \square

The next results characterize the resolvent set and the linear integral manifolds.

Proposition 2.3. *The resolvent set $\rho(A)$ is open. If $\gamma \in \rho(A)$ and $J \subset \rho(A)$ is an interval containing γ , then*

$$\mathcal{U}_\eta = \mathcal{U}_\gamma, \quad \mathcal{V}_\eta = \mathcal{V}_\gamma \quad \text{for all } \eta \in J.$$

Proof. For $\gamma \in \rho(A)$, by definition $\dot{x} = (A(t) - \gamma I)x$ admits a nonuniform exponential dichotomy with an invariant projection $P(t)$. So there exist $K \geq 1$, $\alpha < 0$, $\beta > 0$ and $\mu, \nu \geq 0$ with $\alpha + \mu < 0$ and $\beta - \nu > 0$ such that

$$\begin{aligned} \|\Phi_\gamma(t, s)P(s)\| &\leq K e^{\alpha(t-s)+\mu|s|} \text{ for } t \geq s, \\ \|\Phi_\gamma(t, s)(I - P(s))\| &\leq K e^{\beta(t-s)+\nu|s|} \text{ for } t \leq s. \end{aligned}$$

Set $0 < \sigma < \min\{(\beta - \nu)/2, -(\alpha + \mu)/2\}$. For $\eta \in (\gamma - \sigma, \gamma + \sigma)$, it is easy to see that $P(t)$ is an invariant projection of the evolution operator $\Phi_\eta(t, s) = e^{-\eta(t-s)}\Phi(t, s)$ of system $\dot{x} = (A(t) - \eta I)x$. Moreover we have

$$\begin{aligned} \|\Phi_\eta(t, s)P(s)\| &= e^{(\gamma-\eta)(t-s)}\|\Phi_\gamma(t, s)P(s)\| \leq K e^{(\gamma-\eta+\alpha)(t-s)+\mu|s|} \text{ for } t \geq s, \\ \|\Phi_\eta(t, s)(I - P(s))\| &= e^{(\gamma-\eta)(t-s)}\|\Phi_\gamma(t, s)(I - P(s))\| \leq K e^{(\gamma-\eta+\beta)(t-s)+\nu|s|} \text{ for } t \leq s. \end{aligned}$$

It follows from the choice of σ and η that $\alpha^* = \gamma - \eta + \alpha \leq \alpha^* + \mu < 0$ and $\beta^* = \gamma - \eta + \beta \geq \beta^* - \nu > 0$. This proves that $\dot{x} = (A(t) - \eta I)x$ admits a nonuniform exponential dichotomy for all $\eta \in (\gamma - \sigma, \gamma + \sigma)$, and consequently $(\gamma - \sigma, \gamma + \sigma) \subset \rho(A)$. This proves that $\rho(A)$ is an open set.

For $\eta \in (\gamma - \sigma, \gamma + \sigma)$, the above proof shows that systems $\dot{x} = (A(t) - \eta I)x$ and $\dot{x} = (A(t) - \gamma I)x$ both admit the nonuniform exponential dichotomy with the same invariant projection $P(t)$. By Proposition 2.2 it holds that $\mathcal{U}_\eta = \mathcal{U}_\gamma = \text{Im}P$ and $\mathcal{V}_\eta = \mathcal{V}_\gamma = \text{Ker}P$.

For any given $\gamma^* \in J$, without loss of generality we assume that $\gamma^* \leq \gamma$. For each $\eta \in [\gamma^*, \gamma]$ there exists an open set $(\eta - \sigma_\eta, \eta + \sigma_\eta) \subset J$ with $\sigma_\eta > 0$ such that $\mathcal{U}_\zeta = \mathcal{U}_\eta$ and $\mathcal{V}_\zeta = \mathcal{V}_\eta$ for $\zeta \in (\eta - \sigma_\eta, \eta + \sigma_\eta)$. Since this kind of intervals cover $[\gamma^*, \gamma]$, we get that $\mathcal{U}_\gamma = \mathcal{U}_{\gamma^*}$ and $\mathcal{V}_\gamma = \mathcal{V}_{\gamma^*}$. By the arbitrariness of $\gamma^* \in J$ we can finish the proof of the proposition. \square

Let $\gamma_1, \gamma_2 \in \rho(A)$. By Proposition 2.1 \mathcal{U}_{γ_2} and \mathcal{V}_{γ_1} are both linear integral manifolds. The following result characterizes their intersection.

Proposition 2.4. *For $\gamma_1, \gamma_2 \in \rho(A)$ and $\gamma_1 < \gamma_2$, set $\mathcal{W} = \mathcal{U}_{\gamma_2} \cap \mathcal{V}_{\gamma_1}$. The following conditions are equivalent.*

- (a) $\mathcal{W} \neq \mathbb{R} \times \{0\}$; (b) $[\gamma_1, \gamma_2] \cap \Sigma(A) \neq \emptyset$; (c) $\dim \mathcal{U}_{\gamma_1} < \dim \mathcal{U}_{\gamma_2}$; (d) $\dim \mathcal{V}_{\gamma_1} > \dim \mathcal{V}_{\gamma_2}$.

Proof. The equivalence between (c) and (d) follows easily from Proposition 2.2.

The condition (c) implies (a). Since $(\mathcal{U}_{\gamma_2} \cup \mathcal{V}_{\gamma_1}) \setminus (\mathcal{U}_{\gamma_2} \cap \mathcal{V}_{\gamma_1}) \subset \mathbb{R} \times \mathbb{R}^n$ and $\mathcal{U}_{\gamma_1} < \mathcal{U}_{\gamma_2}$, we have

$$\dim \mathcal{W} = \dim(\mathcal{U}_{\gamma_2} \cap \mathcal{V}_{\gamma_1}) \geq \dim \mathcal{U}_{\gamma_2} + \dim \mathcal{V}_{\gamma_1} - n > \dim \mathcal{U}_{\gamma_1} + \dim \mathcal{V}_{\gamma_1} - n = 0.$$

So $\mathcal{W} \neq \mathbb{R} \times \{0\}$. This proves (a).

The condition (a) implies (b). By contradiction we have $[\gamma_1, \gamma_2] \subset \rho(A)$. So, it follows from Propositions 2.3 and 2.2 that

$$\mathcal{U}_{\gamma_2} \cap \mathcal{V}_{\gamma_1} = \mathcal{U}_{\gamma_1} \cap \mathcal{V}_{\gamma_1} = \mathbb{R} \times \{0\}.$$

This is in contradiction with (a), and consequently (b) follows.

The condition (b) implies (c). If not, since $\mathcal{U}_{\gamma_1} \subseteq \mathcal{U}_{\gamma_2}$ by Proposition 2.2 we have $\dim \mathcal{U}_{\gamma_1} = \dim \mathcal{U}_{\gamma_2}$. It follows that $\dim \mathcal{U}_{\gamma_1}(\tau) = \dim \mathcal{U}_{\gamma_2}(\tau)$ for all $\tau \in \mathbb{R}$. But $\mathcal{U}_{\gamma_1}(\tau)$ and $\mathcal{U}_{\gamma_2}(\tau)$ are linear subspaces of \mathbb{R}^n , we must have $\mathcal{U}_{\gamma_1}(\tau) = \mathcal{U}_{\gamma_2}(\tau)$, and consequently $\mathcal{U}_{\gamma_1} = \mathcal{U}_{\gamma_2}$. By the equivalence of (c) and (d) we also have $\mathcal{V}_{\gamma_1} = \mathcal{V}_{\gamma_2}$. This implies via Proposition 2.2 that the nonuniform exponential dichotomies of $\dot{x} = (A(t) - \gamma_1 I)x$ and $\dot{x} = (A(t) - \gamma_2 I)x$ involve the same invariant projection $P(t)$. So there exist $K_i \geq 1$, $\alpha_i < 0$, $\beta_i > 0$ and $\mu_i, \nu_i \geq 0$

with $\alpha_i + \mu_i < 0$ and $\beta_i - \nu_i > 0$ for $i = 1, 2$ such that

$$\begin{aligned}\|\Phi_{\gamma_i}(t, s)P(s)\| &\leq K_i e^{\alpha_i(t-s)+\mu_i|s|} \text{ for } t \geq s, \\ \|\Phi_{\gamma_i}(t, s)(I - P(s))\| &\leq K_i e^{\beta_i(t-s)+\nu_i|s|} \text{ for } t \leq s.\end{aligned}$$

For $\gamma \in [\gamma_1, \gamma_2]$, take $\alpha = \gamma_1 - \gamma + \alpha_1$, $\beta = \gamma_2 - \gamma + \beta_2$, $\mu = \mu_1$, $\nu = \nu_2$ and $K = \max\{K_1, K_2\}$, we have

$$\begin{aligned}\|\Phi_\gamma(t, s)P(s)\| &= e^{(\gamma_1 - \gamma)(t-s)} \|\Phi_{\gamma_1}(t, s)P(s)\| \leq K e^{\alpha(t-s)+\mu|s|} \text{ for } t \geq s, \\ \|\Phi_\gamma(t, s)(I - P(s))\| &= e^{(\gamma_2 - \gamma)(t-s)} \|\Phi_{\gamma_2}(t, s)(I - P(s))\| \leq K e^{\beta(t-s)+\nu|s|} \text{ for } t \leq s.\end{aligned}$$

This proves that $\gamma \in \rho(A)$ and consequently $[\gamma_1, \gamma_2] \subset \rho(A)$, a contradiction with the assumption (b). Hence (c) holds. We complete the proof of the proposition. \square

2.2 Proof of Theorem 1.1

(a) By Proposition 2.3 $\Sigma(A)$ is closed. We now prove that the number of spectral intervals is no more than n .

Since $\Sigma(A) \subset \mathbb{R}$ is closed, it is either empty or consists of m closed intervals with vanishing intersection. By contradiction we assume that $m > n$. Set $\Sigma(A) = [a_1, b_1] \cup \dots \cup [a_n, b_n] \cup \dots \cup [a_m, b_m]$ with $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_n \leq b_n < \dots < a_m \leq b_m$. Remark that we have the possibility with either $a_1 = -\infty$, or $b_m = \infty$, or both of them. If it is the case, for instance $a_1 = -\infty$ we take $[a_1, b_1]$ as $(-\infty, b_1]$. Choose $\gamma_i \in (b_i, a_{i+1})$ for $i = 1, \dots, n$, we have the linear integral manifolds \mathcal{U}_{γ_i} and \mathcal{V}_{γ_i} for $i = 1, \dots, n$.

From Proposition 2.4 we get that

$$\dim \mathcal{U}_{\gamma_1} < \dim \mathcal{U}_{\gamma_2} < \dots < \dim \mathcal{U}_{\gamma_n} \leq n.$$

So we must have either $\dim \mathcal{U}_{\gamma_1} = 0$ or $\dim \mathcal{U}_{\gamma_n} = n$.

If $\dim \mathcal{U}_{\gamma_1} = 0$, i.e. $\mathcal{U}_{\gamma_1} = \mathbb{R} \times \{0\}$, it follows from Proposition 2.2 that $\mathcal{V}_{\gamma_1} = \mathbb{R} \times \mathbb{R}^n$, and the invariant projection $P(t) = 0$. By the definition of the nonuniform exponential dichotomy we can prove easily that $\dot{x} = (A(t) - \gamma I)x$ for all $\gamma < \gamma_1$ admits a nonuniform exponential dichotomy with the invariant projection $P(t)$. This verifies that $(-\infty, \gamma_1] \subset \rho(A)$. We are in contradiction with the choice of γ_1 .

If $\dim \mathcal{U}_{\gamma_n} = n$, i.e. $\mathcal{U}_{\gamma_n} = \mathbb{R} \times \mathbb{R}^n$, Proposition 2.2 shows that the invariant projection is $P(t) = I$. Then working in a similar way to the proof of the case $\dim \mathcal{U}_{\gamma_1} = 0$, we can prove that $(\gamma_n, \infty) \subset \rho(A)$, a contradiction with the choice of γ_n . Hence we must have $m \leq n$. This proves statement (a).

(b) First we claim that $\dim \mathcal{W}_i \geq 1$ for $i = 1, \dots, m$.

We now prove this claim. For $i = 1$, if $a_1 \neq -\infty$ then $\gamma_0, \gamma_1 \in \rho(A)$ and $[\gamma_0, \gamma_1] \cap \Sigma(A) \neq \emptyset$. Proposition 2.4 shows that $\mathcal{U}_{\gamma_1} \supsetneq \mathcal{U}_{\gamma_0}$. Since $\mathcal{U}_{\gamma_0} \oplus \mathcal{V}_{\gamma_0} = \mathbb{R} \times \mathbb{R}^n$, we must have

$\mathcal{W}_1 = \mathcal{U}_{\gamma_1} \cap \mathcal{V}_{\gamma_0} \supsetneq \mathcal{U}_{\gamma_0} \cap \mathcal{V}_{\gamma_0}$. This implies that $\dim \mathcal{W}_1 \geq 1$ because \mathcal{W}_1 is a linear integral manifold.

If $a_1 = -\infty$ then $\mathcal{W}_1 = \mathcal{U}_{\gamma_1}$ because $\mathcal{V}_{\gamma_0} = \mathbb{R} \times \mathbb{R}^n$. By contradiction we assume that $\dim \mathcal{W}_1 = 0$, i.e. $\mathcal{W}_1 = \mathbb{R} \times \{0\}$. Then $P(t) = 0$ is the invariant projection associated with the nonuniform exponential dichotomy of $\dot{x} = (A(t) - \gamma_1 I)x$. From the proof of (a) we get that $(-\infty, \gamma_1] \subset \rho(A)$. It is in contradiction with the choice of γ_1 . So we have $\dim \mathcal{W}_1 \geq 1$.

For $i > 1$, we have $\gamma_{i-1}, \gamma_i \in \rho(A)$ and $[\gamma_{i-1}, \gamma_i] \cap \Sigma(A) \neq \emptyset$. By Proposition 2.4 we have $\mathcal{U}_{\gamma_i} \supsetneq \mathcal{U}_{\gamma_{i-1}}$. It follows that $\mathcal{W}_i = \mathcal{U}_{\gamma_i} \cap \mathcal{V}_{\gamma_{i-1}} \supsetneq \mathcal{U}_{\gamma_{i-1}} \cap \mathcal{V}_{\gamma_{i-1}}$ and consequently $\dim \mathcal{W}_i \geq 1$. This proves the claim.

Next we claim that $\mathcal{V}_{\gamma_i} = \mathcal{W}_{i+1} + \mathcal{V}_{\gamma_{i+1}}$ for $i = 0, 1, \dots, m-1$. In fact, it follows from the fact that $\mathcal{V}_{\gamma_i} = \mathcal{V}_{\gamma_i} \cap (\mathcal{U}_{\gamma_{i+1}} + \mathcal{V}_{\gamma_{i+1}}) = \mathcal{V}_{\gamma_i} \cap \mathcal{U}_{\gamma_{i+1}} + \mathcal{V}_{\gamma_{i+1}} = \mathcal{W}_{i+1} + \mathcal{V}_{\gamma_{i+1}}$, where in the second equality we have used the fact that $\mathcal{V}_{\gamma_i} \supset \mathcal{V}_{\gamma_{i+1}}$.

Applying the last claim we have

$$\begin{aligned} \mathbb{R} \times \mathbb{R}^n &= \mathcal{U}_{\gamma_0} + \mathcal{V}_{\gamma_0} = \mathcal{W}_0 + \mathcal{V}_{\gamma_0} = \mathcal{W}_0 + \mathcal{W}_1 + \mathcal{V}_{\gamma_1} = \dots \\ &= \mathcal{W}_0 + \mathcal{W}_1 + \dots + \mathcal{W}_m + \mathcal{V}_{\gamma_m} = \mathcal{W}_0 + \mathcal{W}_1 + \dots + \mathcal{W}_m + \mathcal{W}_{m+1}. \end{aligned}$$

Finally for $0 \leq i < j \leq m+1$ we have $\mathcal{W}_i \cap \mathcal{W}_j \subset \mathcal{U}_{\gamma_i} \cap \mathcal{V}_{\gamma_{j-1}} \subset \mathcal{U}_{\gamma_i} \cap \mathcal{V}_{\gamma_i} = \mathbb{R} \times \{0\}$. This proves that $\mathbb{R} \times \mathbb{R}^n = \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \dots \oplus \mathcal{W}_{m+1}$ and consequently statement (b).

We complete the proof of the theorem. \square

2.3 Proof of Theorem 1.2

By the assumption the evolution operator $\Phi(t, s)$ of system (1.1) has a nonuniformly bounded growth, i.e. there exist $K \geq 1$, $a \geq 0$ and $\varepsilon \geq 0$ such that

$$\|\Phi(t, s)\| \leq K e^{a|t-s|+\varepsilon|s|}, \quad t, s \in \mathbb{R}. \quad (2.2)$$

First we claim that $\Sigma(A) \subset [-a - 2\varepsilon, a + 2\varepsilon]$, and so it is bounded.

For $\gamma > a + 2\varepsilon$, we get from (2.2) that

$$\|\Phi_\gamma(t, s)\| \leq K e^{(-\gamma+a)(t-s)+\varepsilon|s|}, \quad \text{for } t \geq s.$$

Since $-\gamma + a + \varepsilon < -\varepsilon \leq 0$, system $\dot{x} = (A(t) - \gamma I)x$ admits a nonuniform exponential dichotomy with the invariant projection $P(t) = I$. This shows that $\gamma \in \rho(A)$ and consequently $(a + 2\varepsilon, \infty) \subset \rho(A)$.

For $\gamma < -a - 2\varepsilon$, we have

$$\|\Phi_\gamma(t, s)\| \leq K e^{-(\gamma+a)(t-s)+\varepsilon|s|}, \quad \text{for } t \leq s.$$

Since $-\gamma - a - \varepsilon > \varepsilon \geq 0$, system $\dot{x} = (A(t) - \gamma I)x$ admits a nonuniform exponential dichotomy with the invariant projection $P(t) = 0$. Hence we have $(-\infty, -a - 2\varepsilon) \subset \rho(A)$. Consequently $\Sigma(A) \subset [-a - 2\varepsilon, a + 2\varepsilon]$. The claim follows.

Next we prove that $\Sigma(A) \neq \emptyset$. The above proof implies that for $\gamma > a + 2\varepsilon$, $\mathcal{U}_\gamma = \text{Im}P = \mathbb{R} \times \mathbb{R}^n$ and $\mathcal{V}_\gamma = \text{Ker}P = \mathbb{R} \times \{0\}$ because $P(t) = I$, and that for $\gamma < -a - 2\varepsilon$, $\mathcal{U}_\gamma = \text{Im}P = \mathbb{R} \times \{0\}$ and $\mathcal{V}_\gamma = \text{Ker}P = \mathbb{R} \times \mathbb{R}^n$ because $P(t) = 0$. Set

$$\gamma^* = \sup\{\gamma \in \rho(A); \mathcal{V}_\gamma = \mathbb{R} \times \mathbb{R}^n\}.$$

Then $\gamma^* \in [-a - 2\varepsilon, a + 2\varepsilon]$. Moreover we have $\gamma^* \in \Sigma(A)$. Otherwise, by Proposition 2.3 there exists a neighborhood J of γ^* such that $J \subset \rho(A)$ and for any $\gamma \in J$ we have $\mathcal{V}_\gamma = \mathcal{V}_{\gamma^*}$. This is in contradiction with the definition of γ^* . So $\Sigma(A) \neq \emptyset$. This proves statement (a).

Let $\Sigma(A) = [a_1, b_1] \cup \dots \cup [a_m, b_m] \subset [-a - 2\varepsilon, a + 2\varepsilon]$ with $m \geq 1$ and $-\infty < a_1 \leq b_1 < \dots < a_m \leq b_m < \infty$. Then statement (b) follows from Theorem 1.1, i.e.

$$\mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \dots \oplus \mathcal{W}_m \oplus \mathcal{W}_{m+1} = \mathbb{R} \times \mathbb{R}^n.$$

We complete the proof of the theorem. □

3 Proof of Theorem 1.3

For proving Theorem 1.3 we need some preliminary results, which will be presented in the next subsection.

3.1 Preparation to the proof of Theorem 1.3

Lemma 3.1. *The following statements are equivalent.*

- (a) *Systems (1.1) and (1.4) are nonuniformly kinematically similar via a transformation $x = S(t)y$.*
- (b) *$\Phi_A(t, s)S(s) = S(t)\Phi_B(t, s)$ for all $t, s \in \mathbb{R}$, where Φ_A and Φ_B are the evolution operators of systems (1.1) and (1.4), respectively.*
- (c) *$S(t)$ is a solution of $\dot{S} = A(t)S - SB(t)$.*

Proof. See Lemma 2.1 of [25], [9] and [10]. □

Lemma 3.2. *If systems (1.1) and (1.4) are nonuniformly kinematically similar, then they have the same nonuniform dichotomy spectrum.*

Proof. It follows from statement (b) of Lemma 3.1 and the fact that $S(t)$ is a nonuniformly Lyapunov matrix. For more details, see e.g. Lemma 3.6 of [10]. \square

Lemma 3.3. *Let $P_0 \in \mathbb{R}^{n \times n}$ be a symmetric projection and $X(t) \in \text{GL}_n(\mathbb{R})$ the group of invertible matrix functions in $t \in \mathbb{R}$. Set $Q(t) = P_0 X(t)^T X(t) P_0 + (I - P_0) X(t)^T X(t) (I - P_0)$. Then*

- (a) $Q(t)$ is positively definite and symmetric.
- (b) There exists a unique positively definite and symmetric matrix function $R(t)$ such that $R(t)^2 = Q(t)$ and $P_0 R(t) = R(t) P_0$.
- (c) $S(t) = X(t) R(t)^{-1}$ is invertible and satisfies $S(t) P_0 S(t)^{-1} = X(t) P_0 X(t)^{-1}$ and

$$\|S(t)\| \leq \sqrt{2}, \quad \|S(t)^{-1}\| \leq \sqrt{\|X(t) P_0 X(t)^{-1}\|^2 + \|X(t) (I - P_0) X(t)^{-1}\|^2}$$

Proof. See Lemma A.5 of [25] and Lemma 3.2 of [10]. \square

Lemma 3.4. *Assume that system (1.1) has an invariant projection $P : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ with $P(t) \neq 0, I$. Then there exists a differentiable nonuniform Lyapunov matrix function $S : \mathbb{R} \rightarrow \text{GL}_n(\mathbb{R})$ such that*

$$S(t)^{-1} P(t) S(t) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for all } t \in \mathbb{R}.$$

Proof. Since $P(t)$ is an invariant projection associated with the evolution operator $\Phi(t, s)$ of system (1.1), i.e. $P(t) \Phi(t, s) = \Phi(t, s) P(s)$ for $t, s \in \mathbb{R}$, it forces that $P(t)$ and $P(s)$ for any $t, s \in \mathbb{R}$ are similar and so have the same rank. The fact that $P(t)$ is a projection implies that for any given $s \in \mathbb{R}$ there exists a $T(s) \in \text{GL}_n(\mathbb{R})$ such that

$$T(s) P(s) T(s)^{-1} = \begin{pmatrix} I_{n_1 \times n_1} & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & 0_{n_2 \times n_2} \end{pmatrix} = P_0 \quad \text{for all } s \in \mathbb{R}, \quad (3.1)$$

where $n_1 = \dim \text{Im} P$ and $n_2 = \dim \text{Ker} P$. Applying Lemma 3.3 to $X(t) = \Phi(t, s) T(s)^{-1}$ and P_0 we get a $R(t)$ satisfying $P_0 R(t) = R(t) P_0$ for $t \in \mathbb{R}$. Set $S(t) = \Phi(t, s) T(s)^{-1} R(t)^{-1}$, we have

$$S(t)^{-1} P(t) S(t) = R(t) T(s) P(s) T(s)^{-1} R(t)^{-1} = R(t) P_0 R(t)^{-1} = P_0,$$

where we have used the fact that $\Phi(t, s) \Phi(s, t) = I$ and the invariance of $P(t)$ with respect to $\Phi(t, s)$.

Finally, the fact that $S(t)$ is a nonuniform Lyapunov matrix function follows from the expression of P_0 and statement (c) of Lemma 3.3. For more details, see the proof of Theorem 3.8 of [10]. We complete the proof of the lemma. \square

3.2 Proof of Theorem 1.3

By the assumptions and Theorem 1.2, we have $m \geq 1$, $a_1 > -\infty$, $a_m < \infty$ and $\mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \dots \oplus \mathcal{W}_m \oplus \mathcal{W}_{m+1} = \mathbb{R} \times \mathbb{R}^n$. Moreover it follows from Theorem 1.1 that $\dim \mathcal{W}_i \geq 1$ for $i = 1, \dots, m$.

In what follows we call the open intervals $(b_0, a_1), (b_1, a_2), \dots, (b_{m-1}, a_m)$ and (b_m, a_{m+1}) the *spectral gaps*, where $b_0 = -\infty$ and $a_{m+1} = \infty$. Choose $\gamma_i \in (b_i, a_{i+1})$ for $i = 0, 1, \dots, m$. By Theorem 1.1 we have $\mathcal{W}_0 = \mathcal{U}_{\gamma_0}$ and $\mathcal{W}_{m+1} = \mathcal{V}_{\gamma_m}$. The following proof combines the origin version of this paper and that of Theorem 3.9 of [10].

For any given $\gamma_0 \in (-\infty, a_1)$, since $(-\infty, \gamma_0] \subset \rho(A)$, the system

$$\dot{x} = (A(t) - \gamma_0 I) x,$$

admits a nonuniform exponential dichotomy with an invariant projection \tilde{P}_0 . Then we have

$$\begin{aligned} \|\Phi(t, s)\tilde{P}_0(s)\| &\leq K_1 e^{(\gamma_0 + \alpha_1)(t-s) + \mu_1|s|} \quad \text{for } t \geq s, \\ \|\Phi(t, s)(I - \tilde{P}_0(s))\| &\leq K_1 e^{(\gamma_0 + \beta_1)(t-s) + \nu_1|s|} \quad \text{for } t \leq s, \end{aligned} \quad (3.2)$$

where $K_1 \geq 1$, $\alpha_1 < 0$, $\beta_1 > 0$, $\mu_1, \nu_1 \geq 0$, and $\alpha_1 + \mu_1 < 0$ and $\beta_1 - \nu_1 > 0$.

We claim that system (1.1) is nonuniformly kinematically similar to

$$\dot{y} = \begin{pmatrix} B_0(t) & 0 \\ 0 & B_{11}(t) \end{pmatrix} y \quad (3.3)$$

with $B_0 : \mathbb{R} \rightarrow \mathbb{R}^{n_0 \times n_0}$ and $B_{11} : \mathbb{R} \rightarrow \mathbb{R}^{m_1 \times m_1}$ differentiable, where $n_0 = \dim \text{Im} P_0$ and $m_1 = \dim \text{Ker} P_0$. Moreover $\Sigma(B_0) = \emptyset$ and $\Sigma(B_{11}) = \Sigma(A)$.

Now we prove this claim. By Lemma 3.4 there exists a differentiable nonuniform Lyapunov matrix function $S_0 : \mathbb{R} \rightarrow \text{GL}_n(\mathbb{R})$ such that

$$S_0(t)^{-1} \tilde{P}_0(t) S_0(t) = \begin{pmatrix} I_{n_1 \times n_1} & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & 0_{n_2 \times n_2} \end{pmatrix} = P_0.$$

Define

$$B(t) = S_0(t)^{-1} (A(t) S_0(t) - \dot{S}_0(t)) \quad \text{for } t \in \mathbb{R}.$$

Lemma 3.1 means that system (1.1) is nonuniformly kinematically similar to

$$\dot{y} = B(t)y, \quad (3.4)$$

via the transformation $x(t) = S_0(t)y(t)$, and that $S_0(t)^{-1} \Phi(t, s) S_0(s)$ is a fundamental matrix solution of (3.4).

Set $R(t) = S_0(t)^{-1} \Phi(t, s) T^{-1}$. From the proof of Lemma 3.4 we have $P_0 R(t) = R(t) P_0$. This implies that $R(t)^{-1}$ and $\dot{R}(t)$ both commute with P_0 . Using the fact that $S_0(t)^{-1} \Phi(t, s) S_0(s)$ is a fundamental matrix solution of (3.4), we can prove easily that

$$B(t) = \dot{R}(t) R(t)^{-1} \quad \text{and} \quad P_0 B(t) = B(t) P_0. \quad (3.5)$$

Write $B(t)$ in the block form, i.e.

$$B(t) = \begin{pmatrix} B_0 & C_0 \\ C_{11} & B_{11} \end{pmatrix},$$

where $B_0 : \mathbb{R} \rightarrow \mathbb{R}^{n_0 \times n_0}$, $B_{11} : \mathbb{R} \rightarrow \mathbb{R}^{m_1 \times m_1}$, $C_0 : \mathbb{R} \rightarrow \mathbb{R}^{n_0 \times m_1}$ and $C_{11} : \mathbb{R} \rightarrow \mathbb{R}^{m_1 \times n_0}$. From the expression of P_0 and the second equation of (3.5) we get that $C_0(t) = 0$ and $C_{11}(t) = 0$.

By Lemma 3.2 systems (1.1) and (3.3) have the same nonuniform dichotomy spectrum. Moreover the evolution operator of system (3.3) has the invariant projection P_0 given in (3.1). So we get from (3.1) and (3.2) that $\Sigma(B_0) \subset (-\infty, a_1)$. This implies that $\Sigma(B_0) = \emptyset$. The claim follows.

For $\lambda \in (b_1, a_2)$, system $\dot{y} = (B(t) - \lambda I)y$ admits a nonuniform exponential dichotomy with an invariant projection \tilde{P}_1 . So there exist $\tilde{K}_1, \tilde{\alpha} < 0, \tilde{\beta} > 0, \tilde{\mu}, \tilde{\nu} \geq 0$ with $\tilde{\alpha} + \tilde{\mu} < 0$ and $\tilde{\beta} - \tilde{\nu} > 0$ such that

$$\begin{aligned} \|\Phi_1(t, s)\tilde{P}_1\| &\leq \tilde{K}_1 e^{(\gamma + \tilde{\alpha}_1)(t-s) + \tilde{\mu}_1|s|} \quad \text{for } t \geq s, \\ \|\Phi_1(t, s)(I - \tilde{P}_1)\| &\leq \tilde{K}_1 e^{(\gamma + \tilde{\beta}_1)(t-s) + \tilde{\nu}_1|s|} \quad \text{for } t \leq s, \end{aligned} \tag{3.6}$$

where $\Phi_1(t, s)$ is the evolution operator of system $\dot{y} = B(t)y$.

Since $\Sigma(B_{11}) = \Sigma(A)$, it follows from the last claim that system

$$\dot{y}_1 = B_{11}(t)y_1,$$

is nonuniformly kinematically similar to

$$\dot{z}_1 = \begin{pmatrix} B_1(t) & 0 \\ 0 & B_{22}(t) \end{pmatrix} z_1,$$

via a nonuniformly Lyapunov transformation $y_1 = S_{11}(t)z_1$. Take

$$S_1(t) = \begin{pmatrix} I_{n_1 \times n_1} & 0 \\ 0 & S_{11}(t) \end{pmatrix} S_0(t).$$

Then system (1.1) is nonuniformly kinematically similar to

$$\dot{z} = \tilde{B}_1(t)z, \quad \tilde{B}_1(t) = \begin{pmatrix} B_0(t) & 0 & 0 \\ 0 & B_1(t) & 0 \\ 0 & 0 & B_{22}(t) \end{pmatrix}, \tag{3.7}$$

via the nonuniformly Lyapunov transformation $x(t) = S_1(t)z(t)$. Since the first inequality of (3.6) also holds for all $\gamma \geq a_2$, taking into account equation (3.7) we get that

$\Sigma(\text{diag}(B_0, B_1)) \subset (-\infty, a_2)$. Similarly from the second inequality of (3.6) we have $\Sigma(B_{22}) \subset (b_1, \infty)$. Hence we have $\Sigma(B_1) = [a_1, b_1]$.

According to the above process, we get a nonuniform Lyapunov transformation $x(t) = \tilde{S}(t)w(t)$, which send system (1.1) to

$$\dot{w} = \tilde{B}_{m-1}(t)w, \quad \tilde{B}_{m-1}(t) = \begin{pmatrix} B_0(t) & & & & \\ & B_1(t) & & & \\ & & \ddots & & \\ & & & B_{m-1}(t) & \\ & & & & B_{mm}(t) \end{pmatrix}, \quad (3.8)$$

with $\Sigma(B_0) = \emptyset$, and $\Sigma(B_i) = [a_i, b_i]$ for $i = 1, \dots, m-1$. Take $\gamma_m \in (b_m, \infty)$, system $\dot{w} = (\tilde{B}_{m-1}(t) - \gamma_m I)w$ admits a nonuniform exponential dichotomy with an invariant projection \tilde{P}_m . Using the same arguments as in the above proof, system (1.1) is nonuniformly kinematically similar to system (1.6). Again as in the above proof we get that $\Sigma(\text{diag}(B_0, \dots, B_m)) \subset (-\infty, \gamma_m)$ and $\Sigma(B_{m+1}) \subset (\gamma_m, \infty)$. This implies that $\Sigma(B_m) = [a_m, b_m]$ and $\Sigma(B_{m+1}) = \emptyset$.

Finally, we prove that the order n_i of the matrix $B_i(t)$ in (1.6) is equal to $\dim \mathcal{W}_i$. Since $\mathcal{W}_0 = \mathcal{U}_{\gamma_0}$ for $\gamma_0 \in (-\infty, b_1)$, by Proposition 2.2 it follows that $\dim \text{Im} \tilde{P}_0 = \dim \mathcal{U}_{\gamma_0} = \dim \mathcal{W}_0$. In addition, the order n_0 of $B_0(t)$ is equal to $\dim \text{Im} \tilde{P}_0$. These verify that $n_0 = \dim \mathcal{W}_0$. Note that $\gamma_0 \in (-\infty, a_1)$ and $\gamma_1 \in (b_1, a_2)$, we get from Propositions 2.2 and 2.4 that

$$\dim \text{Im} \tilde{P}_1 = \dim \mathcal{U}_{\gamma_1} = \dim(\mathcal{U}_{\gamma_1} \cap (\mathcal{U}_{\gamma_0} \oplus \mathcal{V}_{\gamma_0})) = \dim \mathcal{W}_0 + \dim \mathcal{W}_1,$$

where we have used the facts that $\mathcal{U}_{\gamma_0} \subset \mathcal{U}_{\gamma_1}$ and $\mathcal{W}_1 = \mathcal{U}_{\gamma_1} \cap \mathcal{V}_{\gamma_0}$. This implies that $n_1 = \dim \mathcal{W}_1$ because $n_0 + n_1 = \dim \text{Im} \tilde{P}_1$. Similarly $n_2 = \dim \mathcal{W}_2$ follows from the facts that

$$n_0 + n_1 + n_2 = \dim \text{Im} \tilde{P}_2 = \dim \mathcal{U}_{\gamma_2} = \dim(\mathcal{U}_{\gamma_2} \cap (\mathcal{U}_{\gamma_1} \oplus \mathcal{V}_{\gamma_1})) = \dim \mathcal{U}_1 + \dim \mathcal{W}_2,$$

where $\gamma_2 \in (b_2, a_2)$. By induction we can prove that $n_i = \dim \mathcal{W}_i$ for $i = 1, \dots, m$. For $\gamma_m \in (b_m, \infty)$ we get from Proposition 2.2 again that

$$n_{m+1} = \dim \text{Ker} \tilde{P}_m = \dim \mathcal{V}_{\gamma_m} = \dim \mathcal{W}_{m+1}.$$

We complete the proof of the theorem. □

4 Proof of Theorem 1.4

To simplify the proof, in the next subsection we first introduce some basic knowledge on the tensor product and then present some necessary preliminary results on the linear operators defined in the space of the vector-valued homogeneous polynomials.

4.1 The tensor product and its applications

Let V_i for $i = 1, \dots, k$ be n_i dimensional real vector spaces and let $V = V_1 \otimes \dots \otimes V_k$ be their tensor product. Then V is an $n_1 \dots n_k$ dimensional real vector space. The following properties on the tensor product can be found in Lemma 5.4.1 of [1], which will be used later on.

Proposition 4.1. *On the tensor product the following statements hold.*

(i) *The splitting $V_1 = \oplus_{i=1}^{p_1} V_i^{(1)}$ and $V_2 = \oplus_{j=1}^{p_2} V_j^{(2)}$ induce the splitting*

$$V_1 \otimes V_2 = \oplus_{(i,j)=(1,1)}^{(p_1,p_2)} V_i^{(1)} \otimes V_j^{(2)}.$$

(ii) *Let $T_i : V_i \rightarrow V_i$ for $i = 1, \dots, 4$ be linear operators defined in the vector spaces V_i of dimension n_i . Then*

$$T_1 \otimes T_2(x \otimes y) = T_1 x \otimes T_2 y \quad \text{for } x \in V_1, y \in V_2,$$

defines a linear operator $T_1 \otimes T_2$ in $V_1 \otimes V_2$. Moreover we have

$$\begin{aligned} (T_1 + T_2) \otimes T_3 &= T_1 \otimes T_3 + T_2 \otimes T_3, \\ T_1 \otimes (T_2 + T_3) &= T_1 \otimes T_2 + T_1 \otimes T_3, \\ (T_1 \otimes T_2) \circ (T_3 \otimes T_4) &= (T_1 \circ T_3) \otimes (T_2 \circ T_4), \\ (T_1 \otimes T_2)^{-1} &= T_1^{-1} \otimes T_2^{-1} \quad \text{if } T_1 \text{ and } T_2 \text{ are invertible,} \\ \det(T_1 \otimes T_2) &= (\det T_1)^{n_2} (\det T_2)^{n_1}, \quad \|T_1 \otimes T_2\| = \|T_1\| \|T_2\|, \end{aligned}$$

where $T_i + T_j$ makes sense if they are defined in the same vector space.

(iii) *If $A = (a_{ij})$ and $B = (b_{ij})$ are the matrix representations of T_1 and T_2 respectively, then $T_1 \otimes T_2$ has the matrix representation*

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1,n_1}B \\ \vdots & \ddots & \vdots \\ a_{n_1,1}B & \cdots & a_{n_1,n_1}B \end{pmatrix},$$

which is called the Kronecker product of A and B .

Proposition 4.2. *If $\Phi_1(t, s)$ and $\Phi_2(t, s)$ are the evolution operators of $\dot{z}_1 = A(t)z_1$ and $\dot{z}_2 = B(t)z_2$ respectively, then $\Phi_1(t, s) \otimes \Phi_2(t, s)$ is the evolution operator of*

$$\dot{z} = (A_1(t) \otimes I_2 + I_1 \otimes A_2(t))z.$$

Now we recall some results related to the linear operators defined in the space of the vector-valued homogeneous polynomials, part of them can be found in Chapter 8 of [1].

Let

$$H_{n,k}(\mathbb{R}^d) = \left\{ f = \sum_{\tau \in \mathbb{Z}_+^n} f_\tau x^\tau; f_\tau \in \mathbb{R}^d, |\tau| = k \right\},$$

be the vector space of homogeneous polynomials in n variables of degree k with their values in \mathbb{R}^d . Then

$$H_{n,k}(\mathbb{R}^d) = H_{n,k}(\mathbb{R}^1) \otimes \mathbb{R}^d.$$

Set $D = \dim(H_{n,k}(\mathbb{R}^1))$, we have $D = \binom{n+k-1}{k}$, and $H_{n,k}(\mathbb{R}^1)$ and \mathbb{R}^D are equivalent, denoted by $H_{n,k}(\mathbb{R}^1) \cong \mathbb{R}^D$. Let (u_1, \dots, u_d) be a basis of \mathbb{R}^d , and let $\{x^\tau; \tau \in \mathbb{Z}_+^n, |\tau| = k\}$ be a basis of $H_{n,k}(\mathbb{R}^1)$. Clearly $\{x^\tau u_i; i = 1, \dots, d, \tau \in \mathbb{Z}_+^n, |\tau| = k\}$ is a basis of $H_{n,k}(\mathbb{R}^d)$. Under this basis $H_{n,k}(\mathbb{R}^d) \cong \mathbb{R}^{Dd}$ via the equivalence

$$f = \sum_{i=1}^d \sum_{|\tau|=k} f_{\tau,i}(t) x^\tau u_i \longrightarrow (f_{\tau,i}) \in \mathbb{R}^{Dd}.$$

For any $n \times n$ matrix $A(t)$ we define a $D \times D$ matrix

$$N(A)_k = \left(N_{\tau\sigma}^{(k)}(A) \right), \quad (Ax)^\tau = \sum_{\sigma \in \mathbb{Z}_+^n, |\sigma|=k} N_{\sigma\tau}^{(k)}(A) x^\sigma, \quad \tau \in \mathbb{Z}_+^n, |\tau| = k.$$

Usually the entries of $N(A)_k$ are nonlinear functions of the entries of A .

Proposition 4.3. *Let A, B be $n \times n$ matrices, and $k \geq 2$. The following statements hold.*

- (i) $\|N(A)_k\| \leq c\|A\|^k$, $N(I_2) = I_1$ and $N(AB)_k = N(B)_k N(A)_k$, where c is independent of A , and I_2 and I_1 are respectively the $n \times n$ and $D \times D$ unit matrices.
- (ii) If A is invertible then $N(A^{-1})_k = (N(A)_k)^{-1}$.
- (iii) If $A(t) = \text{diag}(A_1(t), \dots, A_p(t))$ with $A_i(t) : \mathbb{R} \rightarrow \mathbb{R}^{n_i \times n_i}$ for $i = 1, \dots, p$, then there exists a $D \times D$ permutation matrix P independent of t under which $N(A)_k$ is similar to a block diagonal matrix

$$\text{diag}(N(A)_\tau, \tau \in \mathbb{Z}_+^p, |\tau| = k),$$

with $N(A)_\tau : \mathbb{R} \rightarrow \mathbb{R}^{q_\tau \times q_\tau}$ and $q_\tau = \prod_{i=1}^p \binom{\tau_i + n_i - 1}{\tau_i}$. Moreover we have

$$\|N(A)_\tau(t)\| \leq c \prod_{i=1}^p \|A_i\|^{\tau_i},$$

where c is independent of $A(t)$.

Proof. Statements (i) and (ii) can be found in Lemma 8.1.2 of [1]. The proof of statement (iii) is given in Proposition 5 of [32]. \square

For $k \geq 2$ we define a linear operator

$$\begin{aligned} T(A)_k : \quad H_{n,k}(\mathbb{R}^1) &\longrightarrow H_{n,k}(\mathbb{R}^1) \\ h = \sum_{\tau \in \mathbb{Z}_+^n, |\tau|=k} h_\tau x^\tau &\longrightarrow \frac{\partial h}{\partial x} A x \triangleq T(A)_k(h). \end{aligned}$$

Proposition 8.3.4 of [1] and Proposition 6 of [32] established a relation between the evolution operators $\Phi_{-T(A)_k}(t)$ and $\Phi_A(t)$.

Proposition 4.4. *Let $\Phi_A(t, s)$ be the evolution operator of $\dot{x} = A(t)x$. Then*

$$\Phi_{-T(A)_k}(t, s) = N(\Phi_{-A}(t, s))_k = N(\Phi_A(t, s))_k^{-1}.$$

Now we define the linear operator

$$\begin{aligned} L_k(t) : \quad H_{n,k}(\mathbb{R}^n) &\longrightarrow H_{n,k}(\mathbb{R}^n) \\ h(x) &\longrightarrow A(t)h(x) - \frac{\partial h(x)}{\partial x} A(t)x. \end{aligned} \tag{4.1}$$

Since $H_{n,k}(\mathbb{R}^n) = H_{n,k}(\mathbb{R}^1) \otimes \mathbb{R}^n$, we can write the operator $L_k(t)$ in

$$L_k(t) = I_1 \otimes A - T(A)_k \otimes I_2,$$

where I_1 and I_2 are the unit matrices on $H_{n,k}(\mathbb{R}^1)$ and \mathbb{R}^n , respectively.

By Propositions 4.3 and 4.4, we get from Proposition 8.3.4 of [1] and Proposition 6 of [32] a relation between the evolution operators $\Phi_{L_k}(t)$ and $\Phi_A(t)$.

Proposition 4.5. *Let $\Phi_A(t, s)$ be the evolution operator of $\dot{x} = A(t)x$. Then the following statements hold.*

$$(a) \quad \Phi_{L_k}(t, s) = \Phi_{-T(A)_k}(t, s) \otimes \Phi_A(t, s) = N(\Phi_A(t, s))_k^{-1} \otimes \Phi_A(t, s) = N(\Phi_A(s, t)) \otimes \Phi_A(t, s).$$

(b) *If $A = \text{diag}(A_1(t), \dots, A_p(t))$, then*

$$\begin{aligned} N(\Phi_A(s, t)) \otimes \Phi_A(t, s) &= \text{diag}(N(\Phi_A(s, t))_\tau \otimes \Phi_{A_j}(t, s), \tau \in \mathbb{Z}_+^p, |\tau| = k, j = 1, \dots, p), \\ \|N(\Phi_A(s, t))_\tau\| &\leq c \prod_{i=1}^p \|\Phi_{A_i}(s, t)\|^{\tau_i}, \end{aligned}$$

where c depends only on n, k and the norm.

4.2 Proof of Theorem 1.4

To simplify notations we write system (1.8) in

$$\dot{x} = A(t)x + f(t, x), \tag{4.2}$$

with $A(t) = \text{diag}(A_1(t), \dots, A_m(t))$ and $f(t, x) = (f_1(t, x), \dots, f_m(t, x))^T$, where T denotes the transpose of a matrix. Assume that there exists a near identity formal transformation $x = y + h(t, y)$ under which system (4.2) is transformed into

$$\dot{y} = A(t)y + g(t, y), \quad (4.3)$$

where $g(t, y)$ is a formal series in y . Then $h(t, y)$ should satisfy the following equation

$$\frac{\partial h}{\partial t} = A(t)h(t, y) - \frac{\partial h}{\partial y}A(t)y + f(t, y + h(t, y)) - \frac{\partial h}{\partial y}g(t, y) - g(t, y). \quad (4.4)$$

Set $w(t, y) \sim \sum_{i=2}^{\infty} w_i(t, y)$ with $w \in \{h, f, g\}$, and $w_i(t, y)$ a homogeneous polynomial of degree i in y . Equation (4.4) can be written in

$$\frac{\partial h_k(t, y)}{\partial t} = L_k(t)h_k(t, y) + F_k(t, y) - g_k(t, y), \quad k = 2, 3, \dots \quad (4.5)$$

where $F_k(t, y)$ is a homogeneous polynomial in y of degree k which is a function of h_2, \dots, h_{k-1} obtained from the expansion of $f(t, y + h(t, y)) - \frac{\partial h}{\partial y}g(t, y)$. We note that $F_k(t, y)$ are successively known. Recall that $L_k(t)$ is the linear operator defined in (4.1).

In the base $\{x^\tau u_i; \tau \in \mathbb{Z}_+^p, |\tau| = k, i = 1, \dots, n\}$ of $H_{n,k}(\mathbb{R}^n)$ each homogeneous polynomial $w_k(t, y)$ with $w \in \{h, F, g\}$ is uniquely determined by its coefficients. Let $w_k(t)$ with $w \in \{h, F, g\}$ be a vector-valued function of dimension Dn which is formed by the coefficients of $w_k(t, y)$ in the given base. Then we get from (4.5) that

$$\frac{d}{dt}h_k(t) = L_k(t)h_k(t) + F_k(t) - g_k(t), \quad (4.6)$$

where for simplicity to notations we still use $L_k(t)$ to denote the linear operator acting on $h_k(t)$.

Since $A(t)$ is a block diagonal matrix, by Proposition 4.5 the evolution operator $\Phi_{L_k}(t, s)$ of $\frac{d}{dt}h_k(t) = L_k(t)h_k(t)$ is also a block diagonal matrix. According to the block diagonal form of $\Phi_{L_k}(t, s)$ given in Proposition 4.5 we separate the vector space \mathbb{R}^{Dn} in the direct sum of the subspaces $\mathbb{R}^{q_\tau n_j}$ for $j = 1, \dots, m, \tau \in \mathbb{Z}_+^m, |\tau| = k$, where n_j is the order of the matrix A_j and q_τ is defined in statement (iii) of Proposition 4.3. Correspondingly we have

$$h_k(t) = \bigoplus_{\tau, j} h_k^{(\tau, j)}(t).$$

So system (4.6) can be written in

$$\frac{d}{dt}h_k^{(\tau, j)}(t) = L_k^{(\tau, j)}(t)h_k^{(\tau, j)}(t) + F_k^{(\tau, j)}(t) - g_k^{(\tau, j)}(t), \quad (4.7)$$

with $\tau \in \mathbb{Z}_+^m, |\tau| = k$ and $j = 1, \dots, m$, where $L_k^{(\tau, j)}(t)$ is the diagonal entry of the block diagonal matrix $L_k(t)$.

Furthermore we separate $p_k^{(\tau, j)}(t) = p_{k1}^{(\tau, j)}(t) + p_{k2}^{(\tau, j)}(t)$ with $p \in \{F, g\}$ in such a way that the former is corresponding to those (τ, j) such that $[a_j, b_j] \cap \sum_{i=1}^m \tau_i[a_i, b_i] = \emptyset$ and the

latter is corresponding to those (τ, j) such that $[a_j, b_j] \cap \sum_{i=1}^m \tau_i [a_i, b_i] \neq \emptyset$. According to this decomposition system (4.7) can be decomposed into two subsystems:

$$\frac{d}{dt} h_k^{(\tau, j)}(t) = L_k^{(\tau, j)}(t) h_k^{(\tau, j)}(t) + F_{k1}^{(\tau, j)}(t) - g_{k1}^{(\tau, j)}(t), \quad (4.8)$$

$$\frac{d}{dt} h_k^{(\tau, j)}(t) = L_k^{(\tau, j)}(t) h_k^{(\tau, j)}(t) + F_{k2}^{(\tau, j)}(t) - g_{k2}^{(\tau, j)}(t). \quad (4.9)$$

In the case $[a_j, b_j] \cap \sum_{i=1}^m \tau_i [a_i, b_i] \neq \emptyset$, i.e. for those (τ, j) the nonuniform dichotomy spectrum is resonant, we choose $g_{k2}^{(\tau, j)}(t) = F_{k2}^{(\tau, j)}(t)$, and consequently equation (4.9) has the trivial solution $h_k^{(\tau, j)}(t) = h_{k2}^{(\tau, j)}(t) = 0$.

In the case $[a_j, b_j] \cap \sum_{i=1}^m \tau_i [a_i, b_i] = \emptyset$, i.e. for those (τ, j) the nonuniform dichotomy spectrum is not resonant, we have either $a_j > \tau_1 b_1 + \dots + \tau_m b_m$ or $b_j < \tau_1 a_1 + \dots + \tau_m a_m$. In this case for any value of $g_{k1}^{(\tau, j)}(t)$ equation (4.8) has a unique solution. For simplicity we take $g_{k1}^{(\tau, j)}(t) = 0$. By the variation of constants formula we obtain from (4.8) that $h_k^{(\tau, j)}(t) = h_{k1}^{(\tau, j)}(t)$ with either

$$h_{k1}^{(\tau, j)}(t) = \int_{-\infty}^t \Phi_{L_k}^{(\tau, j)}(t, s) F_{k1}^{(\tau, j)}(s) ds, \quad \text{if } b_j < \tau_1 a_1 + \dots + \tau_m a_m, \quad (4.10)$$

or

$$h_{k1}^{(\tau, j)}(t) = - \int_t^{\infty} \Phi_{L_k}^{(\tau, j)}(t, s) F_{k1}^{(\tau, j)}(s) ds, \quad \text{if } a_j > \tau_1 b_1 + \dots + \tau_m b_m, \quad (4.11)$$

where $\Phi_{L_k}^{(\tau, j)}(t, s)$ is the evolution operator of the linear equation $\frac{d}{dt} h_k^{(\tau, j)}(t) = L_k^{(\tau, j)}(t) h_k^{(\tau, j)}(t)$.

Combining the two cases, we get $g_k^{(\tau, j)} = g_{k2}^{(\tau, j)} = F_{k2}^{(\tau, j)}$ and $h_k^{(\tau, j)} = h_{k1}^{(\tau, j)}$. Since the integrals in (4.10) and (4.11) are improper, we need to prove that they are convergent and so the functions in (4.10) and (4.11) are well defined.

Proposition 4.6. *Let $k \in (3, \sigma/\varrho)$, where $\sigma = \min\{-\alpha_i, \beta_i; i = 1, \dots, m\}$ and $\varrho = \max\{\mu_i, \nu_i; i = 1, \dots, m\}$. The following statements hold.*

(a) *If $b_j < \tau_1 a_1 + \dots + \tau_m a_m$ we have for $2 \leq r < 2k - 4$*

$$\|h_{r1}^{(\tau, j)}(t)\| \leq C_{r, \tau, j} \exp\left(-\frac{r-1}{2}(2k-r-4)\varrho|t|\right), \quad (4.12)$$

where $C_{r, \tau, j}$ is a positive constant.

(b) *If $a_j > \tau_1 b_1 + \dots + \tau_m b_m$, then $h_{r1}^{(\tau, j)}(t)$ satisfies the same estimation as that given in (4.12) for $2 \leq r < 2k - 4$ with probably the coefficient $C_{r, \tau, j}$ different.*

Proof. From Proposition 4.5 we have

$$\Phi_{L_k}^{(\tau,j)}(t, s) = N(\Phi_A(s, t))_\tau \otimes \Phi_{A_j}(t, s),$$

and

$$\|\Phi_{L_k}^{(\tau,j)}(t, s)\| = \|N(\Phi_A(s, t))_\tau\| \|\Phi_{A_j}(t, s)\| \leq c_k \prod_{i=1}^m \|\Phi_{A_i}(s, t)\|^{\tau_i} \|\Phi_{A_j}(t, s)\|, \quad (4.13)$$

where c_k depends only on k, n and the norm.

Under the nonresonant conditions we define

$$D_{j\tau} = \begin{cases} \tau_1 a_1 + \dots + \tau_m a_m - b_j & \text{if } b_j < \tau_1 a_1 + \dots + \tau_m a_m, \\ a_j - \tau_1 b_1 - \dots - \tau_m b_m & \text{if } a_j > \tau_1 b_1 + \dots + \tau_m b_m. \end{cases}$$

Set $\varepsilon_{j\tau} = D_{j\tau}/2(|\tau| + 1)$. For $\sigma_j \in [a_j - \varepsilon_{j\tau}, a_j)$ and $\rho_i \in (b_j, b_j + \varepsilon_{j\tau}]$, we can check that

$$\begin{aligned} \rho_j - \tau_1 \sigma_1 - \dots - \tau_m \sigma_m &\leq -\frac{1}{2} D_{j\tau} & \text{if } b_j < \tau_1 a_1 + \dots + \tau_m a_m, \\ \sigma_j - \tau_1 \rho_1 - \dots - \tau_m \rho_m &\geq \frac{1}{2} D_{j\tau} & \text{if } a_j > \tau_1 b_1 + \dots + \tau_m b_m. \end{aligned}$$

So systems $\dot{z} = (A_i(t) - \sigma_i I)z$ and $\dot{z} = (A_i(t) - \rho_i I)z$ admit nonuniform exponential dichotomies. Hence there exist $K_i \geq 1$, $\alpha_i < 0$, $\beta_i > 0$, and $\mu_i, \nu_i \geq 0$ with $\alpha_i + \mu_i < 0$ and $\beta_i - \nu_i > 0$ such that

$$\begin{aligned} \|\Phi_{A_i}(t, s)\| &\leq K_i e^{\rho_i(t-s)} e^{\alpha_i(t-s) + \mu_i|s|} & \text{for } t \geq s, \\ \|\Phi_{A_i}(t, s)\| &\leq K_i e^{\sigma_i(t-s)} e^{\beta_i(t-s) + \nu_i|s|} & \text{for } t \leq s. \end{aligned} \quad (4.14)$$

Combining (4.13) and (4.14) we obtain that

$$\|\Phi_{L_k}^{(\tau,j)}(t, s)\| \leq \begin{cases} K_{k,\tau,j} e^{\left[\left(\rho_j - \sum_{i=1}^m \tau_i \sigma_i\right) + \left(\alpha_j - \sum_{i=1}^m \tau_i \beta_i\right)\right](t-s) + \mu_j|s| + \sum_{i=1}^m \tau_i \nu_i|t|} & \text{for } t \geq s, \\ K_{k,\tau,j} e^{\left[\left(\sigma_j - \sum_{i=1}^m \tau_i \rho_i\right) + \left(\beta_j - \sum_{i=1}^m \tau_i \alpha_i\right)\right](t-s) + \nu_j|s| + \sum_{i=1}^m \tau_i \mu_i|t|} & \text{for } t \leq s, \end{cases} \quad (4.15)$$

where $K_{k,\tau,j} = c_k \prod_{i=1}^m K_i^{\tau_i} K_j$. To simplify the notation, for $b_j < \tau_1 a_1 + \dots + \tau_m a_m$ we set

$$\omega_{\tau,j} = \rho_j - \sum_{i=1}^m \tau_i \sigma_i + \alpha_j - \sum_{i=1}^m \tau_i \beta_i.$$

Statement (a). We prove this statement by induction. For $r = 2$, by the assumptions of the theorem we have

$$\|F_2^{(\tau,j)}(t)\| \leq \|p_2(t)\| \leq d_2 e^{-k\varrho|t|}.$$

Recall that $p_2(t)$ is the coefficient vector of the vector-valued homogeneous polynomial $\tilde{f}_2(t, y)$ in the Taylor expansion of $f(t, y)$. Then it follows from (4.10) and (4.15) that for $b_j < \tau_1 a_1 + \dots + \tau_m a_m$ we have

$$\|h_{21}^{(\tau,j)}(t)\| \leq K_{2,\tau,j} d_2 \int_{-\infty}^t e^{\omega_{\tau,j}(t-s) + \mu_j|s| + \sum_{i=1}^m \tau_i \nu_i|t| - k\varrho|s|} ds,$$

where $|\tau| = 2$.

If $t \leq 0$, we have

$$\begin{aligned} \|h_{21}^{(\tau,j)}(t)\| &\leq K_{2,\tau,j}d_2e^{\left(\omega_{\tau,j}-\sum_{i=1}^m\tau_i\nu_i\right)t}\int_{-\infty}^te^{-(\omega_{\tau,j}+\mu_j-k\rho)s}ds \\ &= \frac{K_{2,\tau,j}d_2}{k\rho-\omega_{\tau,j}-\mu_j}e^{\left(k\rho-\sum_{i=1}^m\tau_i\nu_i-\mu_j\right)t} \\ &\leq \frac{K_{2,\tau,j}d_2}{k\rho-\omega_{\tau,j}-\mu_j}e^{(k-3)\rho t} \leq \frac{2K_{2,\tau,j}d_2}{D_{j\tau}}e^{(k-3)\rho t}, \end{aligned}$$

where we have used the facts that $|\tau| = 2$, $\rho = \max\{\mu_i, \nu_i; i = 1, \dots, m\}$ and $k\rho - \omega_{\tau,j} - \mu_j > -(\rho_j - \sum_{i=1}^m \tau_i \sigma_i) \geq \frac{1}{2}D_{j\tau}$.

If $t > 0$, we have

$$\begin{aligned} \|h_{21}^{(\tau,j)}(t)\| &\leq K_{2,\tau,j}d_2e^{\left(\omega_{\tau,j}+\sum_{i=1}^m\tau_i\nu_i\right)t}\left(\int_{-\infty}^0e^{-(\omega_{\tau,j}+\mu_j-k\rho)s}ds + \int_0^te^{-(\omega_{\tau,j}-\mu_j+k\rho)s}ds\right) \\ &= K_{2,\tau,j}d_2e^{\left(\omega_{\tau,j}+\sum_{i=1}^m\tau_i\nu_i\right)t} \times \\ &\quad \left(\frac{1}{k\rho-\omega_{\tau,j}-\mu_j} + \frac{-1}{\omega_{\tau,j}-\mu_j+k\rho}\left(e^{-(\omega_{\tau,j}-\mu_j+k\rho)t}-1\right)\right) \\ &\leq \frac{-K_{2,\tau,j}d_2}{\omega_{\tau,j}-\mu_j+k\rho}e^{-\left(k\rho-\mu_j-\sum_{i=1}^m\tau_i\nu_i\right)t} \leq \frac{2K_{2,\tau,j}d_2}{D_{j\tau}}e^{-(k-3)\rho t}. \end{aligned}$$

We should mention that in the third inequality we have used the facts that $|\tau| = 2$, $k\rho + \alpha_j \leq k\rho - \sigma < 0$ and $\omega_{\tau,j} - \mu_j + k\rho \leq -\frac{1}{2}D_{j\tau} < 0$. In the second inequality we have used the fact $1/(k\rho - \omega_{\tau,j} - \mu_j) + 1/(\omega_{\tau,j} - \mu_j + k\rho) = 2(k\rho - \mu_j)/(k\rho - \omega_{\tau,j} - \mu_j)(\omega_{\tau,j} - \mu_j + k\rho) < 0$, because $k\rho - \mu_j > \rho - \mu_j \geq 0$. This proves statement (a) for $r = 2$.

In order for using induction we assume that statement (a) holds for $r < 2k - 5$. Consider the case $r + 1$. By the assumptions of the theorem and the construction of $F_{r+1}^{(\tau,j)}(t)$ there exists a constant $b_{r+1,\tau,j}$ such that

$$\|F_{r+1}^{(\tau,j)}(t)\| \leq b_{r+1,\tau,j}e^{-\left(rk - \frac{(r-1)(r+4)}{2}\right)\rho|t|}. \quad (4.16)$$

In fact, $F_l^{(\tau,j)}(t)$ is the coefficient of the monomial y^τ in the j th component of the vector-valued homogeneous polynomial $F_l(t, y)$ in y of degree l , and

$$F_l(t, y) = \left[\sum_{r=2}^l f_r \left(t, y + \sum_{s=2}^{l-1} h_s(t, y) \right) \right]_l - \sum_{r=2}^{l-1} \frac{\partial h_r(t, y)}{\partial y} g_{l+1-r}(t, y), \quad (4.17)$$

where $[A(t, y)]_l$ denotes the homogeneous part of degree l of a polynomial function $A(t, y)$ in y . The expression of $F_l(t, y)$ follows from the construction of the transformation $x = y + h(t, y)$ which sent system (4.2) to its normal form (4.3). Recall that f_r , h_r and g_r are

the vector-valued homogeneous polynomials of degree r in y of the Taylor expansions of f, h and g , respectively. Since $g_r(t, y) = F_r(t, y) = F_{r2}(t, y)$ and $h_r(t, y) = h_{r1}(t, y)$, so the estimation (4.16) can be obtained from (4.17) using the induction through the estimations on the coefficients of h_s, g_s for $s = 2, \dots, r$ and (1.10) (i.e. the estimation on the coefficients of f_s) for $s = 2, \dots, r + 1$.

Now from (4.10), (4.15) and (4.16) we get that

$$\|h_{r+1,1}^{(\tau,j)}(t)\| \leq d_{r+1,\tau,j} \int_{-\infty}^t e^{\omega_{\tau,j}(t-s) + \mu_j|s| + \sum_{i=1}^m \tau_i \nu_i |t| - \left(rk - \frac{(r-1)(r+4)}{2}\right)\varrho|s|} ds.$$

where $|\tau| = r + 1$ and $d_{r+1,\tau,j} = K_{r+1,\tau,j} b_{r+1,\tau,j}$.

If $t \leq 0$, working in a similar way to the proof of the case $r = 2$ and by direct integrating we get that

$$\begin{aligned} \|h_{r+1,1}^{(\tau,j)}(t)\| &\leq \frac{d_{r+1,\tau,j}}{(rk - (r-1)(r+4)/2)\varrho - u_j - \omega_{\tau,j}} e^{\left((rk - \frac{(r-1)(r+4)}{2})\varrho - \mu_j - \sum_{i=1}^n \tau_i \nu_i\right)t} \\ &\leq \frac{2d_{r+1,\tau,j}}{D_{\tau,j}} e^{\left(rk - \frac{r(r+5)}{2}\right)\varrho t}, \end{aligned}$$

where we have used the fact that $-(\mu_j + \sum_{i=1}^n \tau_i \nu_i)t \leq -(r+2)t$.

If $t > 0$, we have

$$\begin{aligned} \|h_{r+1,1}^{(\tau,j)}(t)\| &\leq d_{r+1,\tau,j} e^{\left(\omega_{\tau,j} + \sum_{i=1}^m \tau_i \nu_i\right)t} \left(\int_{-\infty}^0 e^{-\left(\omega_{\tau,j} + \mu_j - \left(rk - \frac{(r-1)(r+4)}{2}\right)\varrho\right)s} ds \right. \\ &\quad \left. + \int_0^t e^{-\left(\omega_{\tau,j} - \mu_j + \left(rk - \frac{(r-1)(r+4)}{2}\right)\varrho\right)s} ds \right) \\ &= d_{r+1,\tau,j} e^{\left(\omega_{\tau,j} + \sum_{i=1}^m \tau_i \nu_i\right)t} \left(\frac{1}{\left(rk - \frac{(r-1)(r+4)}{2}\right)\varrho - \omega_{\tau,j} - \mu_j} \right. \\ &\quad \left. + \frac{-1}{\omega_{\tau,j} - \mu_j + \left(rk - \frac{(r-1)(r+4)}{2}\right)\varrho} \left(e^{-\left(\omega_{\tau,j} - \mu_j + \left(rk - \frac{(r-1)(r+4)}{2}\right)\varrho\right)t} - 1 \right) \right) \\ &\leq \frac{-d_{r+1,\tau,j}}{\omega_{\tau,j} - \mu_j + \left(rk - \frac{(r-1)(r+4)}{2}\right)\varrho} e^{-\left((rk - \frac{(r-1)(r+4)}{2})\varrho - \mu_j - \sum_{i=1}^m \tau_i \nu_i\right)t} \\ &\leq \frac{2d_{r+1,\tau,j}}{D_{j\tau}} e^{-\left(rk - \frac{r(r+5)}{2}\right)\varrho t}, \end{aligned}$$

where we have used the fact that $1/((rk - (r-1)(r+4)/2)\varrho - \omega_{\tau,j} - \mu_j) + 1/(\omega_{\tau,j} - \mu_j + (rk - (r-1)(r+4)/2)\varrho) < 0$, because $|\tau| = r + 1 > 2$ and $\alpha_j - \sum_{i=1}^m \tau_i \beta_i + (rk - (r-1)(r+4)/2)\varrho < -(r+2)\sigma + rk\varrho < 0$. Also we have used the fact that $\omega_{\tau,j} - \mu_j + (rk - (r-1)(r+4)/2)\varrho < 0$.

$4)/2)\varrho \leq -\frac{1}{2}D_{j\tau} + \alpha_j - \sum_{i=1}^m \tau_i \beta_i - \mu_j + (rk - (r-1)(r+4)/2)\varrho < -\frac{1}{2}D_{j\tau}$ because $\mu_j \geq 0$ and $\alpha_j - \sum_{i=1}^m \tau_i \beta_i + (rk - (r-1)(r+4)/2)\varrho \leq -(r+2)\sigma + (rk - (r-1)(r+4)/2)\varrho \leq -k\varrho(r+2) + (rk - (r-1)(r+4)/2)\varrho = -2k\varrho - (r-1)(r+4)\varrho/2 < 0$. This proves statement (a) for $r+1$. So by induction, statement (a) follows.

Statement (b). Its proof can be got from (4.15) and from the same arguments as those given in the proof of statement (a). The details are omitted. We finish the proof of the proposition. \square

By Proposition 4.6, the functions $h_{r1}^{(\tau,j)}(t)$ in (4.10) and (4.11) for $r = 2, \dots, 2k-5$ are well defined and bounded. Set $x = y + h(t, y)$ with

$$h(t, y) = \sum_{r=2}^{2k-5} \left(\sum_{|\tau|=r, 1 \leq j \leq m} \bigoplus h_{r1}^{(\tau,j)} y^\tau \right).$$

Then $h(t, y)$ is a polynomial of degree $2k-5$ with the coefficients all bounded functions in $t \in \mathbb{R}$. Hence, by the previous constructions we get that system (1.8) is transformed into (1.11) via the time dependent change of variables $x = y + h(t, y)$.

We complete the proof the theorem. \square

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